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Information theoretical approach to noisy dynamics

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Abstract. The information theoretical structure of a noisy one-dimensional dynamics is investigated. The framework of mutual information and the information flow is given, and the mutual information is calculated in the BZ (Belousov-Zhabotinskii) reaction map and the logistic map. The computational results show the real mechanism of a curious behaviour of noisy dynamics.

1. Introduction

We have found a curious noise effect in a certain class of one-dimensional mappings—noise-induced order (Matsumoto and Tsuda 1983). In our previous papers (Matsumoto and Tsuda 1983, Matsumoto 1983, Tsuda and Matsumoto 1984) on this phenomenon, studies from several directions were attempted. Phenomenologically, noise-induced order appears as the change of the Lyapounov exponent to negative value, the appearance of a sharp peak in the power spectrum, the localisation of orbits and an abrupt decrease of entropy. This phenomenon indicates a difference in the character of the 'randomness' between chaos and noise. Furthermore, this directly stems from a certain kind of non-uniformity character of the strange attractor, which produces non-uniformity of the Markov partition. A modified transition matrix was constructed, the entropy was calculated and was checked with the result of the simulation. It was also indicated that noise can change the dynamics, in other words, there exists hidden dynamics in this class of maps. However, the real cause of this phenomenon was left unresolved.

The purpose of the present paper is to clarify its real mechanism. We introduce new methodology suitable for treating this kind of change of the dynamics. Namely we investigate 'information' to resolve the real cause of the noise-induced order. In § 2, the framework of mutual information and information flow is given. In § 3, using this framework, we investigate the informational aspect of one-dimensional mappings. In § 4, the results are discussed and a scenario of noise-induced order is shown. Section 5 contains a summary.

2. The framework of mutual information and information flow

Let us introduce the Kullback information

$$I_k(p_i) = \sum_i p_i \log(p_i/q_i), \tag{2.1}$$

where p_i is the probability distribution and q_i the standard distribution on a discrete space. This measures the amount of information contained in p relative to q. Usually q is taken as an invariant density or uniform density.

The difference of the amount of information the probability distribution changes p_i to p_i' can be expressed by the difference of the Kullback information. Particularly in the case of one-dimensional mappings, this difference of the Kullback information gives the change in the amount of information contained in the initial distribution. Taking a continuous limit (now $I_k(p)=\int p(x)\log(p(x)/q(x))\,\mathrm{d}x$) this difference can be expressed by

$$I_k(p)-I_k(Fp), \tag{2.2}$$

where F is the Frobenius-Perron operator of a map f:

$$Fp(x)=\sum_{f^{-1}(x)=y}\frac{p(y)}{|\mathrm{d}f(y)/\mathrm{d}y|}.$$

Taking an appropriate set of densities $\{p_i\}$ which sum to the invariant density $p_0(x)$, one can obtain the average difference of the Kullback information by summing the difference for each p_i

$$\sum_i (I_k(p_i)-I_k(Fp_i))$$
$$=\sum_i\int p_i(x)\log\frac{p_i(x)}{q(x)}\,\mathrm{d}x-\sum_i\int Fp_i(x)\log\frac{Fp_i(x)}{q(x)}\,\mathrm{d}x. \tag{2.3}$$

This can be called the information flow (Shaw 1981). Here let us take $p_i(x)=p_0(x)\chi_i(x)$, where $\chi_i(x)$ is a characteristic function of one element of an appropriate partition. Then we have the following expression

$$\begin{aligned}\sum_i (I_k(p_i)-I_k(Fp_i))
&=\sum_i\int p_i(x)\log\frac{p_i(x)}{q(x)}\,\mathrm{d}x-\sum_i\int Fp_i(x)\log\frac{Fp_i(x)}{q(x)}\,\mathrm{d}x\\
&\quad-\left(\int p_0(x)\log\frac{p_0(x)}{q(x)}\,\mathrm{d}x-\int Fp_0(x)\log\frac{Fp_0(x)}{q(x)}\,\mathrm{d}x\right)\\
&=-\sum_i\int Fp_i(x)\log\frac{Fp_i(x)}{q(x)}\,\mathrm{d}x+\sum_i\int Fp_i(x)\log\frac{Fp_0(x)}{q(x)}\,\mathrm{d}x\\
&=\sum_i\int p_0(x)\frac{Fp_i(x)}{Fp_0(x)}\log\left(\frac{Fp_i(x)}{Fp_0(x)}\right)^{-1}\mathrm{d}x.\end{aligned} \tag{2.4}$$

Putting $\rho_i(x)=Fp_i(x)/Fp_0(x)$, we have

$$\sum_i (I_k(p_i)-I_k(Fp_i))=\int p_0(x)\sum_i\rho_i(x)\log\frac{1}{\rho_i(x)}\,\mathrm{d}x \tag{2.5}$$

$\rho_i(x)$ is the probability of the previous point of x being in the ith element of the partition. If the map is one-to-one on each element of the partition, the right-hand side of (2.5) is precisely the amount of information required to map a point inversely. Therefore, this quantity is the amount of information of initial condition lost per iteration. This equals the Lyapounov exponent if $p_0(x)$ is absolutely continuous: $\sum_i (I_k(p_i)-I_k(Fp_i))=\int p_0(x)\log|\mathrm{d}f/\mathrm{d}x|\,\mathrm{d}x$ $(\equiv\lambda)$.

One can measure the amount of information transmitted between two places by the mutual information I. Suppose the signal i is sent from one place, and at the other place the signal j (not necessarily i) is received. Let $p(i, j)$ be the joint probability of its occurrence and the conditional probability $p(j/i) = p(i, j)/p(i)$. Then the mutual information is defined by

$$I(i;j) = \sum_i p(i) \log \frac{1}{p(i)} - \sum_{i,j} p(i)p(j/i) \log \frac{1}{p(j/i)}. \tag{2.6}$$

Let us consider the relation between the transmitted information and the lost information. Divide the unit interval into, say, 100 equal segments and let $p_0(i)$ be the probability of finding the orbital point in the ith segment. Then $p_0(j/i)$ is the transition probability under the mapping from i to j. We obtain approximately the expression $F(\chi_i(x)) \simeq p_0(j/i)\chi_j(x)$. Substituting this expression into (2.5), we obtain

$$\begin{aligned}(I_k(p_i) - I_k(Fp_i)) &\simeq \sum_i p(i) \log \frac{p(i)}{q(i)} - \sum_{i,j} p(i)p(j/i) \log \frac{p(i)p(j/i)}{q(i)} \\ &= \sum_{i,j} p(i)p(j/i) \log \frac{1}{p(j/i)}. \end{aligned} \tag{2.7}$$

Therefore, the second term of the right-hand side of (2.6) is just the expression (2.7). If the partition becomes finer, the first term of the right-hand side of (2.6) becomes larger, but the second term remains *nearly* equal to the information loss (equation (2.5)).

It is convenient to consider a 'computer register' (figure 1) in order to explain the difference between the mutual information and the information flow. The above partition corresponds to observing this register only through the highest, say four, places. The mutual information accounts for all information escaping from this window. But the information flow has a sign according to its direction, and only its average is a matter of concern. The information loss is equal to the information flow. Only in the case when the fluctuation of information flow is small do we obtain the equality in expression (2.7). In other words, one can see the fluctuation of information flow in terms of the mutual information.

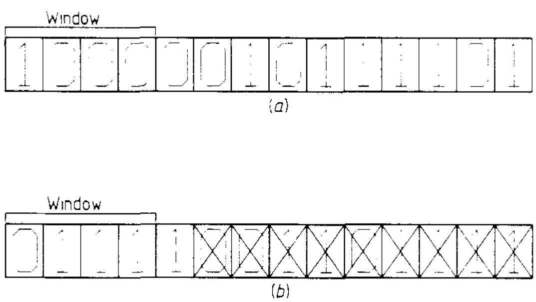

Figure 1. Computer register: (a) indicates the noise-free case and (b) the noisy case. The window corresponds to the width of the observation.

3. Calculation method and mutual information

Our window of observation here is the partition of the unit interval into 100 equal segments. Then a condition of the system is specified by the number i of the segment in which the system is found. The amount of information gained about the initial condition when we know the condition after n interations is given by $I_n(i;j)$:

$$I_n(i;j) = \sum_i p_0(i) \log \frac{1}{p_0(i)} - \sum_{i,j} p_0(i) p_n(j/i) \log \frac{1}{p_n(j/i)}, \tag{3.1}$$

where $p_n(j/i)$ is the conditional probability of a point started from the ith segment falling in the jth segment after n iterations. In the present calculation, we start with 10 000 points uniformly distributed over each segment of the partition and calculate $p_n(j/i)$. We calculate $I_n(i;j)$ for the *BZ map* and the *logistic map* with and without noise. The results are shown in figure 2(*a*)-(*d*). The following three points are of importance:

(i) the linear decrease in the logistic map,

(ii) the exponential decrease in the BZ map,

(iii) the humps in the noise-free BZ map.

We discuss these points in the next section.

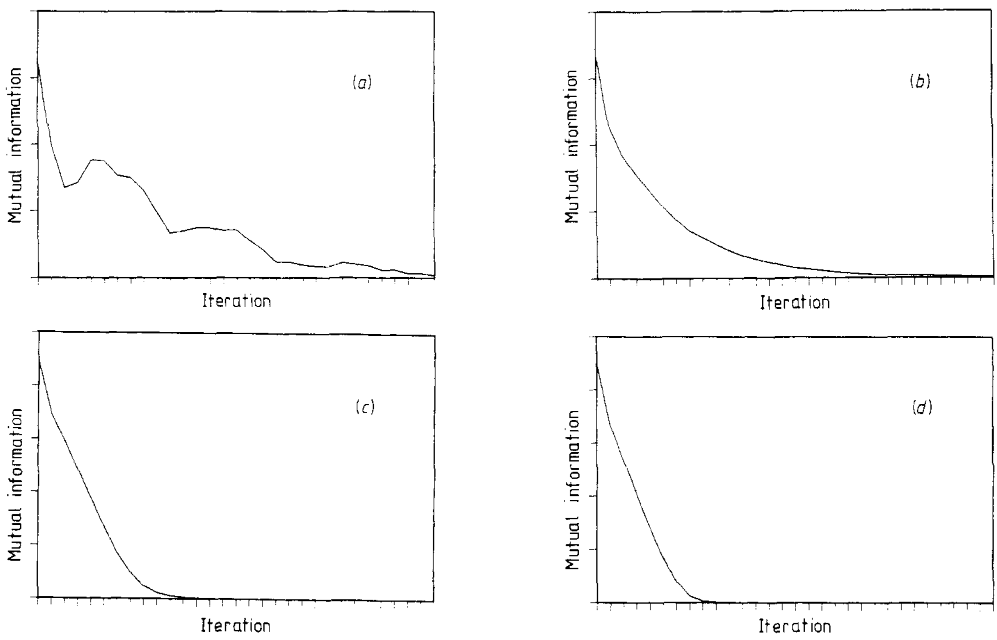

Figure 2. Mutual information: (*a*) and (*b*) indicate the noiseless and the noisy case respectively in the BZ map. (*c*) and (*d*) show the noiseless and the noisy case respectively in the logistic map.

4. Discussion

(i) The linear decrease corresponds to a simple picture of the information flow, namely the average information flow consists of a monotonic flow directed to the left-hand side of the register in figure 1. Therefore, the mutual information loses as much as the information flow carries away, in other words, (2.7) is correct. We have the expression $I_n = I_0 - \lambda n$, where λ is the Lyapounov exponent, since the information flow rate is equal to the Lyapounov exponent. This situation is essentially the same as in the Bernoulli map ($X_{n+1} = 2X_n \pmod 1$).

(ii) The situation is very different in this case. Clearly, (2.7) does not hold. The average information flow directed to the left-hand side of the register is a sum of contributions from the right-directed flow as well as the left-directed flow.

As a result it is conceivable that the information contained in a place of the register spreads over the whole register in subsequent iterations. On the other hand, the information loss occurs at the largest scale (i.e. at the left end of the register) as seen in (2.5). From these two effects, the mutual information decreases at the same rate in any time, namely, $I_n = I_0 r^n$. As the information loss equals the Lyapounov exponent, $I_0(1-r) = \lambda$ holds. This relation does not assert that $I_0 - I_1 = \lambda$. As mentioned previously, the right-directed flow additionally carries away the information in the window. Thus r must be obtained from fitting the whole curve of mutual information.

(iii) In the case without noise, the information that has flowed to the right-hand side of the register must return to the window, since the information loss mechanism does not exist there. This is the reason why the hump appears in the curve of mutual information as shown in figure 2(a). When the noise is on, the additional information loss mechanism works near the right end of the register. So the hump disappears, as shown in figure 2(b).

We checked the relation between mutual information and the Lyapounov exponent in several cases. Now, the mechanism of noise-induced order is apparent. A significant amount of information about the initial condition is erased by the noise in maps like the BZ, for the information carried by the right-directed flow is caught by the noise and destroyed. This implies a drastic change in the dynamics. For maps of logistic type this is not the case: the information of the initial condition flows to the left end of the register where noise cannot reach. The information flow is safe from the noise. The existence of the phenomenon 'noise-induced order' is clearly checked from the feature of the curve of mutual information.

In the case of periodic orbits, the initial information is consumed at the right end of the register. In this respect, the BZ map with noise is the intermediate case between chaos and periodicity.

5. Summary and outlook

There are two types of decay of information about the initial condition. One is a linear decay and the other is an exponential decay. In the former case, the initially localised information remains localised in subsequent time. In the latter case, the initially localised information spreads considerably and information mixing occurs. Furthermore, in the latter case, the chaos itself is *fragile*, because extended information is easily destroyed by noise.

One cannot overlook the important application of the present result to the memory mechanism of the brain. It is known that the *holographic memory* plays an important role in the brain (Kohonen 1984, Pribram 1971). Our result—the exponential decay of the information—clearly indicates that *a part of the whole information is contained in each place of the register.* This shows the possibility to give a concrete mechanism of the holographic memory of the brain. One should note that this holographic character of the information appears only in the system with hidden dynamics, i.e. implicated order. In the brain, the maps of the BZ type (not of the logistic type) might exist.

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